USA Mathematical Talent Search

Solutions to Problem 1/3/17

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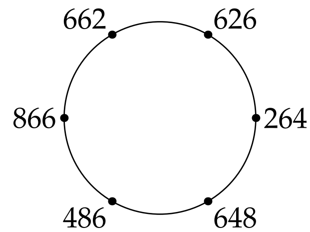
1/3/17. For a given positive integer n, we wish to construct a circle of six numbers as shown at right so that the circle has the following properties:



(a) The six numbers are different three-digit numbers, none of whose digits is a 0.

(b) Going around the circle clockwise, the first two digits of each number are the last two digits, in the same order, of the previous number.

(c) All six numbers are divisible by n.

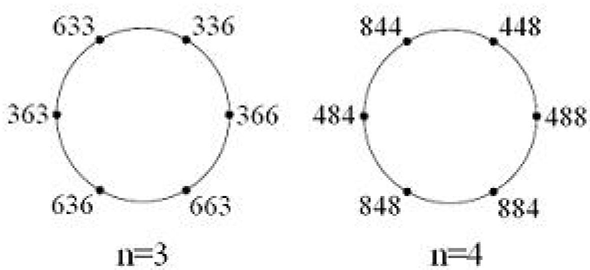
The example above shows a successful circle for $n = 2$. For each of $n = 3$, 4, 5, 6, 7, 8, 9, either construct a circle that satisfies these properties, or prove that it is impossible to do so.

Credit This problem was based on a proposal by George Berzsenyi, founder of the USAMTS.

Comments First, you must determine for each given n whether such a circle of numbers exists or not. When it exists, such a circle is not hard to find. When it does not exist, well-known divisibility rules of numbers can be used to give a rigorous proof. *Solutions edited by Naoki Sato.*

Solution 1 by: Shotaro Makisumi (10/CA)

Circles can be constructed for $n = 3$, 4, 6, and 7. An example of each is shown below.



We will show that such a construction is impossible for $n = 5$, 8, and 9.

$n = 5$: Each number must end in 0 or 5 for divisibility by 5, but 0 cannot be used, so all numbers must end in 5. Then, going around the circle, the ten digits must also all be 5, as the hundred digits. Thus, all numbers must be 555, which violates rule (a).

$n = 8$: All units digits must be even for divisibility by 8. Then the ten digits and also the hundred digits must all be even. Since 8 divides 200, 8 must also divide the last two digits. The only possibilities are $x24$, $x48$, $x64$, and $x88$, where x is an even digit. Going clockwise around the circle, $x24$ and $x64$ both force $x48$ to be the next number, which then forces $x88$ as the next number, and then 888. Thus, 888 will necessarily be repeated before the circle is complete, violating rule (a).

$n = 9$: Assume such a construction is possible, and pick a number abc (or $100a + 10b + c$) in the cycle. A number is divisible by 9 if and only if the sum of its digits equals a multiple of 9, so $9|(a + b + c)$. If we let the next number clockwise be bcd, then $9|(b + c + d)$, so $9|[(a+b+c)-(b+c+d)]$ or $9|(a-d)$. Since $1 \leq a$, $d \leq 9$, we must have $a = d$, so $bcd = bca$. Continuing clockwise, we see by the same argument that the next numbers are cab and abc. The number abc must then appear twice, which again violates rule (a).