1/2/18. Find all positive integers n such that the sum of the squares of the digits of n is 2006 less than n.

Credit This problem was proposed by Dave Patrick.

Comments The simplest approach in this problem begins with establishing good bounds, which helps to reduce the casework. For example, with the proper bounds, you can show that n must have exactly four digits. *Solutions edited by Naoki Sato.*

Solution 1 by: Carlos Dominguez (10/OH)

We first show that n must have exactly 4 digits. Since the sum of the squares of the digits is positive, $n > 2006$. Now suppose n has d digits with $d \geq 5$. The smallest integer with d digits is 10^{d-1}, and the largest possible sum of the squares of the digits is $9^2 \cdot d = 81d$. We claim that

$$81d < 10^{d-1} - 2006.$$

When $d = 5$, we get $405 < 10000 - 2006 = 7994$, which is true. Now suppose that it holds for $d = k$, for some positive integer $k \geq 5$. Then add 81 to both sides to get

$$81(k+1) < (10^{k-1} + 81) - 2006 < 10^k - 2006,$$

which completes the induction. This implies if $d \geq 5$, then the sum of squares of the digits of n must be less than $n - 2006$. Therefore, n must have exactly 4 digits.

We want to find digits (a, b, c, d), where $n = 1000a + 100b + 10c + d$, such that $a^2 + b^2 + c^2 + d^2 = 1000a + 100b + 10c + d - 2006$. The maximum value of $a^2 + b^2 + c^2 + d^2$ is $9^2 \cdot 4 = 324$, which means that n is at most $324 + 2006 = 2330$. Since we already know that $n > 2006$, the first digit must be $a = 2$.

Substituting into our equation, we get

$$b^2 + c^2 + d^2 = 100b + 10c + d - 10.$$

Case 1: $b = 0$. We have

$$\begin{aligned}
c^2 - 10c + d^2 - d &= -10 \\
\Rightarrow \quad (c-5)^2 + \left(d - \frac{1}{2}\right)^2 &= 25 + \frac{1}{4} - 10 \\
\Rightarrow \quad (2c-10)^2 + (2d-1)^2 &= 61.
\end{aligned}$$

We see that 61 can be written as the sum of two squares only as $6^2 + 5^2$. This gives the solutions $(c, d) = (2, 3)$, (8,3). So two possible values of n are 2023 and 2083. Indeed, $2^2 + 0^2 + 2^2 + 3^2 + 2006 = 2023$ and $2^2 + 0^2 + 8^2 + 3^2 + 2006 = 2083$.

Case 2: $b > 0$. If $b = 1$, then

$$\begin{aligned} c^2 - 10c + d^2 - d &= 89 \\ \Rightarrow \quad (c-5)^2 + \left(d - \frac{1}{2}\right)^2 &= 89 + 25 + \frac{1}{4} \\ \Rightarrow \quad (2c-10)^2 + (2d-1)^2 &= 457. \end{aligned}$$

However, since the left hand side is at most $(2 \cdot 0 - 10)^2 + (2 \cdot 9 - 1)^2 = 389$, equality is impossible. ($(2c-10)^2$ is maximized when c is the the smallest digit 0, whereas $(2d-1)^2$ is maximized when d is the largest digit 9.) For the same reason, the cases $b = 2$ and $b = 3$ are impossible as well.

Therefore, the only answers are 2023 and 2083.