USA Mathematical Talent Search

Solutions to Problem 1/2/17

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1/2/17. At the right is shown a 4×4 grid. We wish to fill in the grid such that each row, each column, and each 2×2 square outlined by the thick lines contains the digits 1 through 4. The first row has already been filled in. Find, with proof, the number of ways we can complete the rest of the grid.

1	2	3	4

Credit This problem was proposed by George Berzsenyi, founder of the USAMTS.

Comments This is a relatively simple counting problem (inspired by the latest Sudoku puzzles and simplified to a 4×4 grid), where you need a little care to make sure that you have covered all the cases and that you haven't counted any grids twice. Lynnelle Ye shows a particularly nice approach. *Solutions edited by Naoki Sato.*

Solution 1 by: Lynnelle Ye (8/CA)

The numbers in the second row must be one of the following: 3 4 1 2, 4 3 2 1, 3 4 2 1, or 4 3 1 2.

In the first two cases, the first and third columns are missing the same numbers, and the second and fourth columns are missing the same numbers, so the number chosen for the third row, first column determines the number chosen for the third row, third column, and the number chosen for the third row, second column determines the number chosen for the third row, fourth column. Once the third row is chosen, the fourth row is determined. For example, with the first case for the second row,

1	2	3	4
3	4	1	2
2 or 4	1 or 3	4 or 2	3 or 1

So these two cases contribute a total of $2^3 = 8$ possibilities.

In the second two cases, the number chosen for the third row, first column determines the number chosen for the third row, either third or fourth column. This then determines the number in the third row, second column, which then determines the third row, whatever column is left. Again, the fourth row is determined. For example, with the third case for the second row and the third row, first column equal to 2,

1	2	3	4
3	4	2	1
2	1	4	3

The third row, fourth column must be 3, which means that the third row, second column must be 1, which means that the third row, third column must be 4. So these two cases contribute a total of $2^2 = 4$ possibilities.

Therefore, there are a total of $8 + 4 = 12$ ways to complete the rest of the grid.