USA Mathematical Talent Search

Solutions to Problem 1/1/17

www.usamts.org

1/1/17. An increasing arithmetic sequence with infinitely many terms is determined as follows. A single die is thrown and the number that appears is taken as the first term. The die is thrown again and the second number that appears is taken as the common difference between each pair of consecutive terms. Determine with proof how many of the 36 possible sequences formed in this way contain at least one perfect square.

Credit This problem was taken from the book "St. Mary's College Mathematics Contest Problems."

Comments This is a straight-forward problem using modular arithmetic, requiring only some basic casework. We would like to point out that technically, the term "quadratic residue" only applies when the modulus is prime, and 0 is not included. For example, the quadratic residues modulo 7 are 1, 2, and 4. Otherwise, the term "square" modulo m should be used. *Solutions edited by Naoki Sato.*

Solution 1 by: Derrick Sund (12/WA)

Note: throughout this problem, I will use (a, b) to denote the infinite arithmetic sequence obtained from first rolling the number a, and then rolling the number b.

It is a well-known fact that if i is a quadratic residue (mod j), there are infinitely many perfect squares congruent to i (mod j), and that if k is not a quadratic residue (mod j), then there are no perfect squares congruent to k (mod j). Thus, if a is a quadratic residue (mod b), then the sequence (a, b) (which consists of all numbers greater than or equal to a which are congruent to a (mod b)) must contain a perfect square, and likewise, if a is not a quadratic residue (mod b), the sequence (a, b) cannot contain a perfect square.

Therefore, the sequence (a, b) will contain a perfect square if and only if a is a quadratic residue (mod b). Since it is also well-known that you can determine all quadratic residues (mod n) simply by squaring all numbers from 1 to n, inclusive, and finding their residues (mod n), we can finish the problem by finding the quadratic residues for mods 2, 3, 4, 5, and 6 ((mod 1) need not be considered, since $(a, 1)$ trivially contains all perfect squares greater than or equal to a).

The quadratic residues (mod 2) are 0 and 1. Therefore, (1,2), (2,2), (3,2), (4,2), (5,2), (6,2) all contain perfect squares.

The quadratic residues (mod 3) are 0 and 1. Therefore, (1,3), (3,3), (4,3), (6,3) all contain perfect squares, while (2,3), (5,3) do not.

The quadratic residues (mod 4) are 0 and 1. Therefore, (1,4), (4,4), (5,4) all contain perfect squares, while (2,4), (3,4), (6,4) do not.

The quadratic residues (mod 5) are 0, 1, and 4. Therefore, (1,5), (4,5), (5,5), (6,5) all contain perfect squares, while (2,5), (3,5) do not.

The quadratic residues (mod 6) are 0, 1, 3, and 4. Therefore, (1,6), (3,6), (4,6), (6,6) all contain perfect squares, while (2,6), (5,6) do not.

Thus, since 6 is the highest number that a die can roll, we have 27 sequences with perfect squares: (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (1,3), (3,3), (4,3), (6,3), (1,4), (4,4), (5,4), (1,5), (4,5), (5,5), (6,5), (1,6), (3,6), (4,6), (6,6).

Solution 2 by: Jeff Nanney (12/TX)

Denote the result of the first die toss d. Denote the result of the second die toss a. Naturally, $a, d \in \mathbb{N}$ such that $1 \leq a, d \leq 6$. We now seek to determine which ordered pairs (a, d) will yield at least one perfect square of the form $a(n-1)+d$, where $n \in \mathbb{N}$. Though a variety of approaches are available, the most natural is to examine the 6 cases according to the values of a. In particular, we will use the basic property that squaring all the members of a residue system yields each possible residue for a perfect square in that modulus. In general, we are seeking to find a solution in positive integers to the equation $an + d = x^2$, which is equivalent to finding for which d there exists some x such that $x^2 \equiv d \pmod{a}$.

1. Let $a = 1$. Thus, for $1 \leq d \leq 6$, we must find some integer x such that $x^2 \equiv d \pmod{1}$. Since all positive integers are congruent modulus 1, we know that all d are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, immediately noting that 9 is a perfect square attainable by all the sequences, regardless of the value of d. Thus, we have 6 sequences so far for which a perfect square is produced.

2. Let $a = 2$. For $1 \leq d \leq 6$, we must find some integer x such that $x^2 \equiv d \pmod{2}$. Because $0^2 = 0$ and $1^2 = 1$, and all members of the residue system are perfect squares, we know that all d are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, immediately noting that 9 is a perfect square attainable when d is odd, and 16 is a perfect square attainable when d is even. Thus, we have 6 more sequences for which a perfect square is produced.

3. Let $a = 3$. For $1 \leq d \leq 6$, we must find some integer x such that $x^2 \equiv d \pmod{3}$. Because $0^2 = 0$, $1^2 = 1$, and $2^2 \equiv 1 \pmod{3}$, we know that $d \equiv 0, 1 \pmod{3}$, or

$d = 1, 3, 4, 6$, are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, noting that 9 is a perfect square attainable when $d \equiv 0 \pmod 3$, and 16 is a perfect square attainable when $d \equiv 1 \pmod 3$. Thus, we have 4 more sequences for which a perfect square is produced.

4. Let $a = 4$. For $1 \leq d \leq 6$, we must find some integer x such that $x^2 \equiv d \pmod 4$. Because $0^2 = 0$, $1^2 = 1$, $2^2 \equiv 0 \pmod 4$, and $3^2 \equiv 1 \pmod 4$, we know that $d \equiv 0, 1 \pmod 4$, or $d = 1, 4, 5$, are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, noting that 16 is a perfect square attainable when $d \equiv 0 \pmod 4$, and 9 is a perfect square attainable when $d \equiv 1 \pmod 4$. Thus, we have 3 more sequences for which a perfect square is produced.

5. Let $a = 5$. For $1 \leq d \leq 6$, we must find some integer x such that $x^2 \equiv d \pmod 5$. Because $0^2 = 0$, $1^2 = 1$, $2^2 \equiv 4 \pmod 5$, $3^2 \equiv 4 \pmod 5$, and $4^2 \equiv 1 \pmod 5$, we know that $d \equiv 0, 1, 4 \pmod 5$, or $d = 1, 4, 5, 6$, are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, noting that 25 is a perfect square attainable when $d \equiv 0 \pmod 5$, 16 is a perfect square attainable when $d \equiv 1 \pmod 5$, and 4 is a perfect square attainable when $d \equiv 4 \pmod 5$. Thus, we have 4 more sequences for which a perfect square is produced.

6. Let $a = 6$. For $1 \leq d \leq 6$, we must find some integer x such that $x^2 \equiv d \pmod 6$. Because $0^2 = 0$, $1^2 = 1$, $2^2 \equiv 4 \pmod 6$, $3^2 \equiv 3 \pmod 6$, $4^2 \equiv 4 \pmod 6$, and $5^2 \equiv 1 \pmod 6$, we know that $d \equiv 0, 1, 3, 4 \pmod 5$, or $d = 1, 3, 4, 6$, are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, noting that 25 is a perfect square attainable when $d \equiv 1 \pmod 6$, 9 is a perfect square attainable when $d \equiv 3 \pmod 6$, 16 is a perfect square attainable when $d \equiv 4 \pmod 6$, and 36 is a perfect square attainable when $d \equiv 0 \pmod 6$. Thus, we have 4 more sequences for which a perfect square is produced.

Combining the conclusions from each of the above 6 cases, we find that of the 36 possible sequences, exactly $6 + 6 + 4 + 3 + 4 + 4 = 27$ sequences contain at least one perfect square.